Math 523, Functional Analysis, Fall, 2014

Meets: Monday-Wednesday-Friday, 2-2:50 PM (Location, to be announced)

Description: Functional Analysis deals roughly with infinite dimensional *normed* (or more generally topological) vector spaces. It has lots of connections with real and complex analysis, differential equations, numerical analysis, combinatorics, and physics (e.g. quantum theory). A norm on a vectorspace V assigns to each vector $v \in V$ a nonnegative "length" $\|v\|$ so that

$$\|v\| = 0 \Leftrightarrow v = 0, \quad \| - v\| = \|v\|, \quad \text{and} \quad \|\mathrm{v} + \mathrm{w}\| \leq \|\mathrm{v}\| + \|\mathrm{w}\| \quad \text{for all } \mathrm{v}, \mathrm{w} \in \mathrm{V} \ .$$

The normed vector space $(V, \| \ \|)$ is called *complete* or simply a *Banach space* if, for any sequence $v_i \in V$ with $\sum_{i=1}^{\infty} \|v_i\| < \infty$, there is a unique point $v \in V$ so that $\|v - \sum_{i=1}^{j} v_i\| \to 0$ as $j \to \infty$. A *Hilbert space* is a Banach space in which $\|v\| =< v, v >^{1/2}$ for some inner product (i.e. positive-definite, symmetric, bilinear function) $<, >$ on V. The existence of such an inner product for a norm $\| \ \|$ is equivalent to the purely geometric equality $\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$ holding for all $v, w \in V$. A finite dimensional Hilbert space is equivalent, by a linear isomorphism to the $\mathbf{R^n}$ with $< v, w >= v \cdot w$. The simplest infinite dimensional Hilbert space is the space of square-summable sequences

$$\ell^2 \ = \ \{(a_1, a_2, a_3, \ldots) \ : \ \sum_{i=1}^{\infty} a_i^2 < \infty\} \quad \text{with} \quad < (\mathrm{a}_1, \mathrm{a}_2, \ldots), (\mathrm{b}_1, \mathrm{b}_2, \ldots) > = \ \sum_{\mathrm{i}=1}^{\infty} \mathrm{a_i b_i} \ .$$

The first part of this course will be a fairly quick treatment of properties of a Hilbert space, most of which are based on *the existence, for any point b outside of a convex set A of a unique closest point a in A, that is, so that* $\|a - b\|$ *is a minimum.* For example, there is the Riesz representation theorem that *every real-valued continuous linear function from V to* $\mathbf{R}$ *is given by taking the inner product with some fixed vector.* These facts will already allow us to connect with numerous approximation results and classical Fourier series. Much deeper questions concern *operators* which are continuous linear functions from the Hilbert space to itself. Here we consider various important classes of operators, including spectral theory for compact operators and some of the interesting applications offered in the text. The last part of the course will concern Banach spaces, with specific examples, some basic general theorems, and some generalizations, where possible, of the Hilbert space results.

We will also describe some applications of functional analysis. These frequently use *spaces of functions* with specific norms and operators which either involve differentiation of functions or, inversely, solving (or approximating solutions of) linear differential equations.

Textbooks:

Avner Friedman, *Foundations of Modern Analysis*, Dover Paperback.

Robert Zimmer, *Essential Results of Functional Analysis*, University of Chicago Paperback.

Other recommended references:

Nicholas Young, *An Introduction to Hilbert Space*, Cambridge University Presss.

Walter Rudin, *Functional Analysis*, McGraw-Hill.

Prerequisites: Linear Algebra, Point-set Topology (at least in $\mathbf{R^n}$), Some beginning analysis.

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Accommodations: If you have a (documented) disability and you would like an accommodation, please speak with me during the first two weeks of class. Any such information will of course remain confidential, and you certainly do not need to tell us any details beyond those we should expect to be hearing from the Office of Disability Support Services describing the accommodations you will be requiring. You'll also have to file forms with Disability Support Services in the Ley Student Center in order to have those accommodations implemented.